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Stochastic multiresonance in a bistable sawtooth potential driven by correlated multiplicative and additive noise

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Abstract. We present an analytic investigation of the signal-to-noise ratio (SNR) by studying the bistable sawtooth system driven by correlated Gaussian white noises. The analytic expression of SNR is obtained. Based on it, we detect the phenomenon of stochastic multiresonance, which arises from the dependence of SNR upon the noises correlation coefficient. Furthermore, there exists not only resonance, but also suppression in the $SNR \sim D$ (the additive noise intensity) curve and the $SNR \sim Q$ (the multiplicative noise intensity) curve.

PACS. 05.40-a Fluctuation phenomena, random processes, noise, and Brownian motion – 02.50.-r Probability theory, stochastic processes, and statistics

1 Introduction

The phenomenon of stochastic resonance (SR) has begun to attract wide attention in a great quantity of scientific fields since the original discovery of Benzi *et al.* [1] and Nicolis *et al.* [2]. Consequently, the theories of SR have developed and improved gradually [3,15]. Based on the transition rate, McNamara *et al.* [5] detailedly studied the SR phenomenon, which occurs in the bistable system, by using the method of adiabatic approximation. Then, Hu *et al.* [10] applied the theory of eigenfunction expansion to get systematic results in terms of the perturbation expansion, and more accurate results were accessible. Further more, Gammaitoni *et al.* [13] proposed a method to investigate SR by means of the residence-time distribution. In this phase, all these works as to stochastic resonance were fastened their attention on the case of unimodal shape, *i.e.*, the SR phenomenon was shown by the appearance of only one maximum in the output SNR. Their interests were immersed in researching the stochastic systems, in which SR occurs, but they ignored to place emphasis on the form of SR itself, a more important aspect of SR. Therefore, none of literature, before 1997, had studied and reported SR of this kind, in which the SNR of the system appears more than one maximum with the variety of the noise intensity, *i.e.*, the phenomenon named stochastic multiresonance.

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In 1997, the phenomenon of stochastic multiresonance was detected firstly by Vilar and Rubi's work [16]. They investigated four different stochastic systems, and the $SNR \sim \ln D$ curves were plotted by means of numerical simulation. As one can see from these curves, the $SNR \sim \ln D$ curves exhibit several maxima, even exhibit maximum periodically.

The stochastic systems were considered with only one noise item in Vilar's work, and the analytic expression of SNR was not accessible. Further, they analyzed the relation between SNR and the noise intensity by virtue of the method of scaling argument, according to the symmetries and invariances of the systems [16,20]. Nevertheless, as a matter of fact, many of stochastic systems were driven by several noise sources. Moreover, in certain situations, the noises may be correlated with each other in many kinds of forms, which was studied by Fulinski *et al.* [17–19] in lots of references. In this paper, we present an analytic investigation of the SNR by studying the bistable sawtooth system driven by correlated Gaussian white noises. The analytic expression of SNR is obtained. Based on it, we detect the phenomenon of stochastic multiresonance, which arises from the dependence of SNR upon the noises correlation coefficient. Furthermore, there exists not only resonance, but also suppression in the $SNR \sim D$ (the additive noise intensity) curve and the $SNR \sim Q$ (the multiplicative noise intensity) curve.

This paper is organized as follows. In Section 2, we present the model and the detailed theoretic derivation of analytic expression of SNR. Section 3 is some conclusions and discussion.

2 Model and theoretic derivation

Consider a one dimensional system driven by correlated multiplicative and additive noise, in which the bistable potential is modulated by a weak periodic signal and both noises are correlated to Gaussian white noises. The model is described by a Stratonovich Langevin equation,

$$\dot{x} = -U^{'}(x) + A(t) + g(x)\xi(t) + \eta(t) \ , \qquad (1)$$

where $U(x)$ is a bistable sawtooth potential, whose expression is assumed as

$$U(x) = \begin{cases} \infty \ , \text{when } -\infty < x < -L \\ bx/L \ , \text{when } -L \leqslant x < 0 \\ -bx/L \ , \text{when } 0 \leqslant x \leqslant L \\ \infty \ , \text{when } L < x < \infty \end{cases} \qquad (2)$$

in which b and L is the height and width of the potential barrier, respectively.

In equation (1), $A(t) = A_0 \cos \Omega t$, is the periodic modulated signal, in which A_0 is the amplitude of the signal and Ω the frequency.

The multiplicative function $g(x)$ is chosen to be piecewise constant,

$$g(x) = \begin{cases} c \ , \text{ when } -L \leqslant x < 0 \\ -c \ , \text{ when } 0 \leqslant x \leqslant L \end{cases} \qquad (3)$$

here, c is a constant.

The noises $\xi(t)$ and $\eta(t)$ in equation (1) are correlated in the following manner

$$\langle \xi(t) \rangle = \langle \eta(t) \rangle = 0,$$

$$\left\langle \xi(t)\xi(t^{'}) \right\rangle = 2Q\delta \qquad (4)$$

$$\left\langle \eta(t)\eta(t^{'}) \right\rangle = 2D\delta(t - t^{'}), \qquad (5)$$

$$\left\langle \xi(t)\eta(t^{'}) \right\rangle = \left\langle \eta(t)\xi(t^{'}) \right\rangle = 2\lambda\sqrt{QD}\delta(t - t^{'}), \ \ |\lambda| \leqslant 1 \qquad (6)$$

here λ, the noises correlation coefficient, denotes the correlative strength between $\xi(t)$ and $\eta(t)$. Q and D are the intensities of the noises.

Above is our concrete model. For the sake of clarity of our thinking, we show the procedure of the investigation by two steps. First step, we assume that $W_{\pm}(t)$ can be made a Taylor expansion as following form [5]

$$W_{\pm}(t) = \frac{1}{2}[W_0 \mp \alpha_1 A_0 \cos \Omega t + O(A_0^2) + \cdots]. \qquad (7)$$

Here $W_+(t)$ and $W_-(t)$ are the nonstationary transition rates from the state x_+ to x_- and from the state x_- to x_+ respectively, where x_+ and x_- are the stationary solution to equation (1) without noises and signal. W_0 is the transition rate without modulated signal, and α_1 is the value of first derivative at the point of Taylor expansion.

According to the potential model described by equation (2), $x_{\pm} = \pm L$ are both stationary solutions to equation (1) without signal and noises. Therefore, $W_{\pm}(t)$ are the nonstationary transition rates from L to $-L$ and from $-L$ to L respectively.

The Fokker-Planck equation corresponding to equation (1) with equations (4–6) is given by [21]

$$\partial_t p(x,t) = -\partial_x \left[-U^{'}(x) + A_0 \cos \Omega t + G(x)G^{'}(x) \right] p(x,t) + \partial_{xx} G^2(x) p(x,t) \ , \qquad (8)$$

where, $G(x) = \sqrt{D}\left[Rg^2(x) + 2\lambda\sqrt{R}g(x) + 1 \right]^{1/2}$, $R = Q/D$.

The mean first passage time (MFPT) corresponding to equation (8), $T(R, \lambda, A(t))$, is determined by [22,23]

$$T(R,\lambda,A) = D^{-1} \int_{-L}^{L} \mathrm{d}x H(x) \exp\left[\frac{\Phi(x)}{D} \right] \times \int_{-\infty}^{x} \mathrm{d}y H(y) \exp\left[-\frac{\Phi(x)}{D} \right], \qquad (9)$$

in which,

$$H(x) = \left[Rg^2(x) + 2\lambda\sqrt{R}g(x) + 1 \right]^{-1/2} , \qquad (10)$$

$$\Phi(x) = \int_{-\infty}^{x} \mathrm{d}y H^2(y) \left[-U^{'}(x) + A_0 \cos \Omega t \right] . \qquad (11)$$

Combining equations (9–11) with equations (2–3), the MFPT is obtained

$$\begin{aligned} T(R,\lambda,A) = & \frac{D}{H_+^2 K_+^2} \left[\exp\left(\frac{LK_+H_+^2}{D} \right) - 1 \right] \\ & + \frac{D}{H_-^2 K_-^2} \left[\exp\left(\frac{LK_-H_-^2}{D} \right) - 1 \right] \\ & + \frac{-DH_-}{H_+K_+(H_-^2K_- - H_+^2K_+)} \\ & \times \left[\exp \frac{L}{D}\left(K_-H_-^2 - K_+H_+^2 \right) - 1 \right] \\ & + \frac{D}{H_+H_-K_+K_-} \exp\left(\frac{LK_+H_+^2}{D} \right) \\ & \times \left[\exp\left(\frac{LK_-H_-^2}{D} \right) - 1 \right] - \frac{1}{K_+} - \frac{1}{K_-} . \end{aligned} \qquad (12)$$

Here,

$$H_{\pm} = \left[c^2R \pm 2\lambda c\sqrt{R} + 1 \right]^{-1/2} ,$$

and

$$K_{\pm} = \pm b/L \ + A_0 \cos \Omega t \ .$$

The analytic expression of the MFPT $T(R,\lambda,A(t))$ is given by equation (12). When $D \ll \Delta U(x)$ (the barrier height), the transition rate is the reciprocal of the MFPT, $T(R,\lambda,A(t))$, by virtue of the result in reference [22–24], *i.e.*, $W(t) = 1/T(R,\lambda,A(t))$. Moreover, according to the expression of transition rate in equation (7), W_0 and α_1 is determined respectively as

$$\frac{1}{2}W_0 = W(t)|_{A=0} = [T(R,\lambda,A)|_{A=0}]^{-1} \equiv T_0^{-1}, \quad (13)$$

$$\frac{1}{2}\alpha_1 = T_0^{-2}\left(\frac{\partial T}{\partial A}\right)\bigg|_{A=0}. \quad (14)$$

When $A(t) = 0$, the case without signal, we obtain

$$\begin{aligned} T_0 \equiv T(R,\lambda,A)|_{A=0} &= \frac{L^2 D}{H_+^2 b^2}\left[\exp\left(\frac{bH_+^2}{D}\right) - 1\right] \\ &+ \frac{L^2 D}{H_-^2 b^2}\left[\exp\left(\frac{-bH_-^2}{D}\right) - 1\right] \\ &+ \frac{DH_- L^2}{H_+(H_+^2 + H_-^2)b^2}\left[\exp\frac{-b}{D}\left(H_+^2 + H_-^2\right) - 1\right] \\ &+ \frac{-DL^2}{H_+ H_- b^2}\exp\left(\frac{bH_+^2}{D}\right)\left[\exp\left(\frac{-bH_-^2}{D}\right) - 1\right]. \end{aligned} \quad (15)$$

According to equation (13), it is easy to know $W_0 = 2/T_0$. Then, combining equation (12) with equation (14), it yields

$$\begin{aligned} \frac{1}{2}\alpha_1 &= \frac{1}{T_0^2}\left(\frac{\partial T}{\partial A}\right)\bigg|_{A=0} \\ &= \frac{1}{T_0^2}\left\{\frac{L^3}{b^2}\exp(\sigma_+) + \frac{L^3}{b^2}\exp(-\sigma_-)\right. \\ &+ \frac{-2DL^3}{H_+^2 b^3}\left[\exp(\sigma_+) - 1\right] + \frac{2DL^3}{H_-^2 b^3}\left[\exp(-\sigma_-) - 1\right] \\ &+ \frac{DH_- L^3}{H_+(H_+^2 - H_-^2)b^3}\left[\exp(\sigma_+ - \sigma_-) - 1\right] \\ &+ \frac{H_- L^3 (H_-^2 - H_+^2)}{H_+(H_+^2 + H_-^2)b^2}\exp(\sigma_+ - \sigma_-) \\ &+ \frac{-H_- L^3}{H_+ b^2}\exp(\sigma_+)\exp(-\sigma_-) \\ &+ \frac{-H_+ L^3}{H_- b^2}\exp(\sigma_+)\left[\exp(-\sigma_-) - 1\right] \\ &\left. + \frac{DH_- L^3 (H_-^2 - H_+^2)}{H_+(H_+^2 + H_-^2)^2 b^3}\left[\exp(\sigma_+ - \sigma_-) - 1\right] + \frac{2L^3}{b^2}\right\}. \end{aligned} \quad (16)$$

where, $\sigma_+ = \frac{bH_+^2}{D}$, and $\sigma_- = \frac{bH_-^2}{D}$. Now, both W_0 and α_1 are known. The second step, in the adiabatic approximation the output power spectral density is given by [5]

$$S(\omega) = S_S(\omega) + S_N(\omega)$$

$$= \frac{\pi L^2 \alpha_1^2 A_0^2}{2(W_0^2 + \Omega^2)}\delta(\omega - \Omega) + \left[1 - \frac{\alpha_1^2 A_0^2}{2(W_0^2 + \Omega^2)}\right]\frac{2L^2 W_0}{W_0^2 + \omega^2}, \quad (17)$$

in which, the signal output spectral density, $S_S(\omega)$, is a δ function at the signal frequency, and the noise output spectrum, $S_N(\omega)$, is a Lorentzian form.

So, the signal-to-noise ratio is given by [5,24]

$$SNR = \frac{P_S}{S_N(\omega = \Omega)} = \frac{P_S}{\left[1 - \frac{\alpha_1^2 A_0^2}{2(W_0^2 + \Omega^2)}\right]\frac{2L^2 W_0}{W_0^2 + \Omega^2}}. \quad (18)$$

where,

$$P_S = \int_0^\infty S_S(\omega)\mathrm{d}\omega = \frac{\pi L^2 \alpha_1^2 A_0^2}{2(W_0^2 + \Omega^2)}.$$

Here, P_S is the output power of signal, and SNR the output signal-to-noise ratio. In this paper, we study SNR by using the theory of adiabatic approximation proposed by McNamara *et al.* in reference [5]. The theory is valid under the conditions: $\Omega \ll 1$, $A_0 \ll 1$ and $D \ll 1$, and for the presence of the multiplicative noise in our model, the limit $Q \ll 1$ is added. Due to these limits, the process of the system reaching local equilibrium in each well is very transient, comparing with the process of the system reaching equilibrium between both wells and the period of the modulating signal. Moreover, it is also the reason that we can only expand to the first order of $A(t)$ in equation (7). In order to keep our results valid, throughout this paper we will restrict all chosen parameters in the case of the adiabatic approximation.

3 Conclusion and discussion

By virtue of the expression for SNR equation (18), we will analyze emphatically the influence to SNR by additive noise intensity D, multiplicative noise intensity Q and correlation coefficient λ between both noises. In order to illustrate the results we plot in Figures 1–3 the dependence of SNR upon the parameters.

(I) Appearance of stochastic multiresonance

In Figure 1, we depict the $SNR \sim \lambda$ curve for four cases. Chosen the additive noise intensity D as the parameter, when D is very small, for $D = 0.001$, there exists a minimum in the $SNR \sim \lambda$ curve (*i.e.* suppression), as shown in Figure 1. When D is added, for $D = 0.08$, there are two peaks in the $SNR \sim \lambda$ curve at which SR occur twice, and the heights of both peaks are different. Hence, this phenomenon was called as stochastic multiresonance, which was rarely found in ordinary system. For $D = 0.20$, there are two peaks, between which the

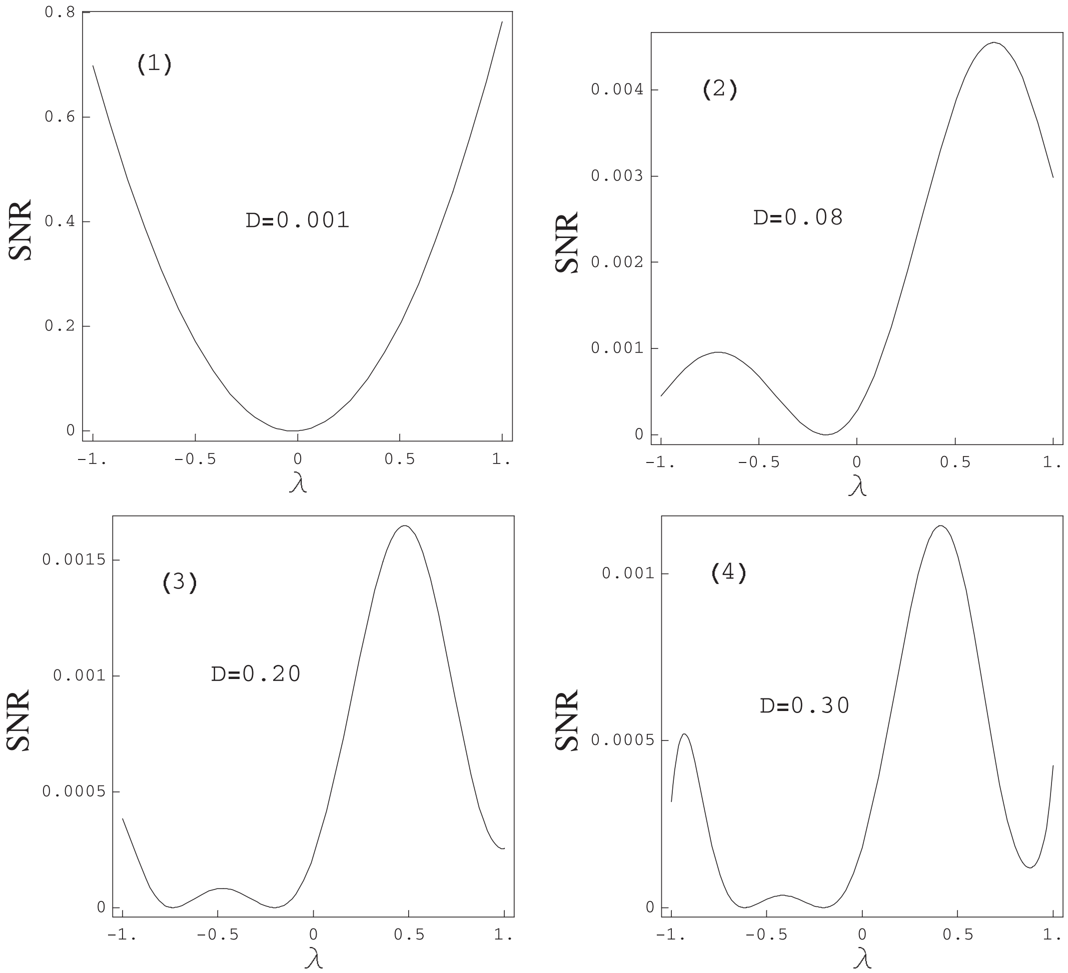

Fig. 1. The output SNR as a function of the correlation coefficient λ, for different values of the additive noise intensity D. The values of the other parameters are $L = 0.1$, $b = 1$, $c = 30$, $Q = 0.001$, $A_0 = 0.1$, $\Omega = 0.001$.

difference of their heights becomes larger, and two minimum in the $SNR \sim \lambda$ curve. Proceed to increase D, for $D = 0.30$, three peaks appear in the $SNR \sim \lambda$ curve, and the height of the peak in the right is the largest, while that of mid the smallest. It is demonstrated that the appearance of stochastic multiresonance not only depends the additive noise intensity, but also closely correlative with the noise correlation coefficient (*i.e.* the correlation strength between stochastic force and potential fluctuation).

(II) Suppression and resonance to SNR by the additive noise intensity

There exist four cases in the $SNR \sim D$ curve, which is shown in Figure 2. Chosen the multiplicative coefficient c as the parameter, when $c = 7.0$, SNR increases monotonously as D, as shown in Figure 2(1). When c is increased, for $c = 18.0$, the curve exhibits a maximum, *i.e.*, the typical SR continues to increase c, when $c = 20.4$, a minimum firstly, then a maximum appears in the $SNR \sim D$ curve, that is to say, it exhibits suppression firstly, SR later. This case is rare in the former works. For $c = 25.0$, SNR is decreasing monotonously with increasing D.

(III) Resonance and suppression to SNR by the multiplicative noise intensity

As shown in Figure 3, there exist four cases in the $SNR \sim Q$ curve, too. Similarly chosen c as the parameter, for $c = 1.7$, SNR increases monotonously as Q. For $c = 4.6$, the typical SR appears, shown in Figure 3(2). For $c = 10.8$, the shape of the $SNR \sim Q$ curve shows reverse case compared with the $SNR \sim D$ curve, because it exhibits SR first, suppression later in the $SNR \sim Q$ curve, shown in Figure 2(3) and Figure 3(3). For $c = 30.0$, SNR is increasing monotonously with increasing Q. It is obvious that the multiplicative coefficient is the key factor to influence SNR.

In summary, the output SNR exhibits SR, not only at only one special value of the noise intensity, but also at several different values. Moreover, there is not only SR, but also suppression in a stochastic system. We believe that stochastic multiresonance is a new form of SR, and it may lead us to explore the essence of SR further. The heights of peaks, which occur in this paper, are different. In order to conform to the practical application, we expect to obtain the peaks of the same height by perfecting our model.

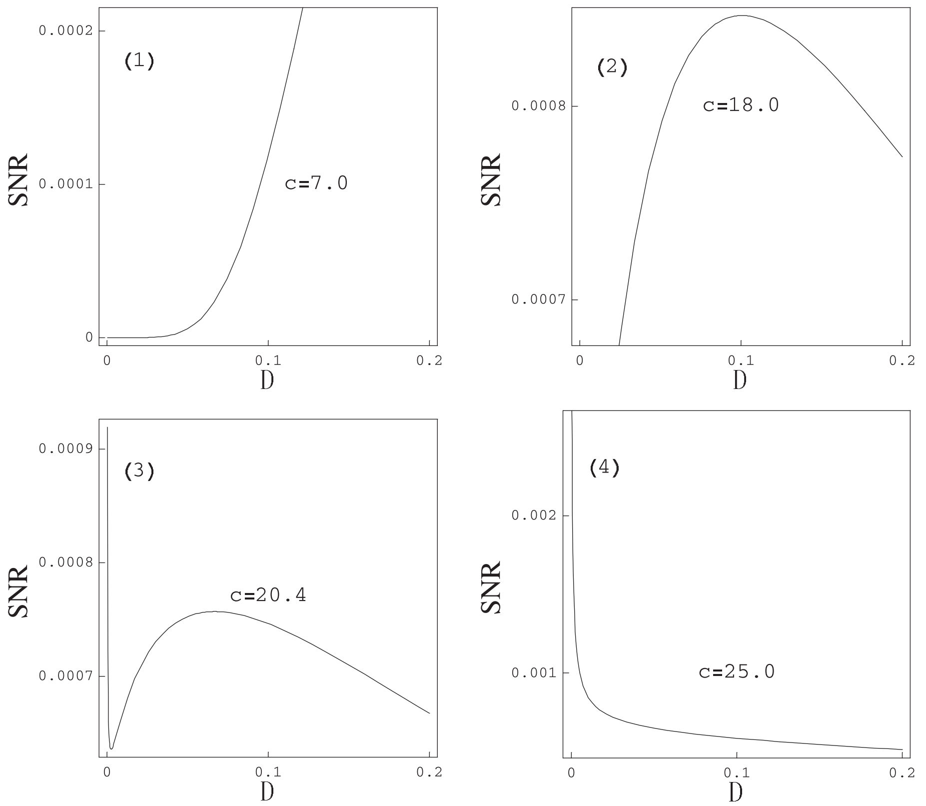

Fig. 2. The output SNR as a function of the additive noise intensity D, for the different values of the multiplicative coefficient c. The values of the other parameters are $L = 0.1$, $b = 2$, $\lambda = 0.5$, $Q = 0.001$, $A_0 = 0.1$, $\Omega = 0.001$.

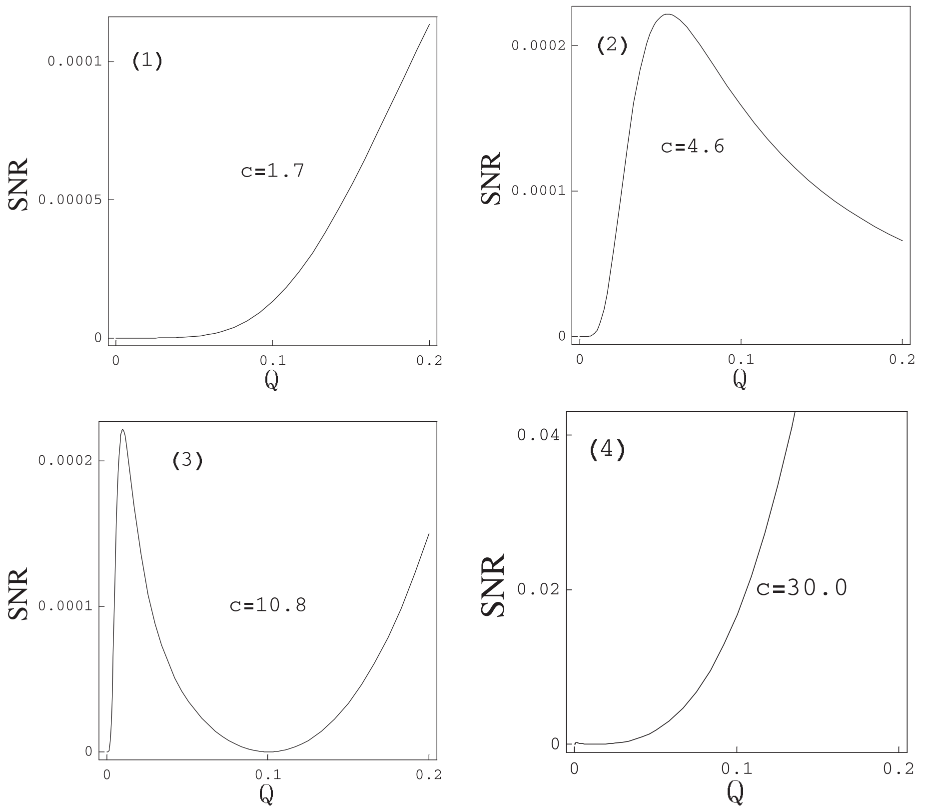

Fig. 3. The output SNR as a function of the multiplicative noise intensity Q, for different values of the multiplicative coefficient c. The values of the other parameters are $L = 0.1$, $b = 5$, $D = 0.2$, $\lambda = 0$, $A_0 = 0.1$, $\Omega = 0.001$.

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